TOPOLOGY AND LINEAR TRANSFORMATION THEORY IN ACTIVE NETWORK SYNTHESIS*

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Linear transformation theory is used to devise a method of parameter matrix construction from a specified rational response function. Realization of this function is achieved through use of topology with a single-element-kind realization technique applied to the parameter matrices thus found. The result offers an approach to the synthesis of active and/or non-bilateral networks since the parameter matrices may be dissymmetrical and pertinent to nonpositive energy forms as well as to passive bilateral networks. Advantage over other methods lies chiefly in the complete generality of topological configurations obtainable and in the controllability of element value distributions.

I. INTRODUCTION

The method of active network synthesis described here, in contrast with other known procedures, has the advantage of being completely general in the sense that no implicit topological configuration or relative orientation of active and passive elements is involved. Thus, the more conventional approach begins by assuming some physical arrangement of tubes or transistors relative to passive network components (such as feedback circuits or the combination of a negative impedance converter with a passive two-terminal pair) and seeks to achieve the desired properties of the response function associated with this arrangement through an appropriate choice of the detailed configuration and element value distribution in the passive portion. Success in such a procedure depends upon a prior knowledge of the potential possibilities afforded by the particular arrangement chosen and of its inherent limitations.

On the other hand, a synthesis procedure that makes no prior commitments regarding topological configuration or arrangement of either passive or active elements, but leaves these matters to be fixed solely by the demands made upon the desired response function together with perfectly general realizability conditions, permits accomplish-

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ment of the full potential achievable from networks composed of such element kinds. It hence deserves to be accorded the title of "active network synthesis," since it possesses the same objective properties that characterize the more familiar procedures designated as "passive network synthesis."

These characteristics of the method described here result primarily from the fact that parameter matrices play an essential part in its implementation. Realization of the resultant network in topological structure and element value distribution is accomplished from parameter matrices rather than from some manipulative procedure applied to the desired rational response function, as is characteristic of the usual realization techniques. Generality and flexibility both stem from making parameter matrices the focal point of this procedure.

Two major problems must be solved, however, in order to accomplish this end: (a) we must develop a way of constructing the pertinent parameter matrices from stated response functions; and (b) we must solve the problem of realizing a network when its parameter matrices are given.

The solution to (a) is based upon the thought that the normal coordinates of a linear system, on the one hand, place in evidence the natural frequencies which characterize the pertinent response functions, and, on the other, are intimately related to the parameter matrices through a linear transformation. The solution to (b) is found by recognizing that all that is needed is the realization technique for a single-element-kind network from its parameter matrix plus a topological scheme for establishing the resultant structure when several parameter matrices are specified. A detailed discussion of these things and some illustrative examples follow.

Although this new approach can be applied to general RLC networks with embedded active devices (such as vacuum tubes, transistors, or "tunnel" diodes), we shall consider here only the somewhat simpler problem of constructing RC networks with such embedded active elements where the object is to achieve driving-point or transfer functions without the pole-zero restrictions associated with ordinary passive RC networks.

II. DETERMINATION OF PARAMETER MATRICES FROM GIVEN RESPONSE FUNCTIONS

We assume a passive RC network that has embedded in it one or more resistive elements that are active and/or non-bilateral; we characterize its equilibrium on a node basis in terms of a capacitance matrix C and a conductance matrix G. While C is symmetrical and its associated quadratic form is positive definite, neither of these conditions is necessarily fulfilled for G. Both matrices are of order n, the number of tree branches in a pertinent graph. If s denotes the complex frequency variable, then the so-called node admittance matrix

is given by

$$Y = Cs + G \ , \tag{1}$$

and the equilibrium equations read

$$Y \times E = I_s \tag{2}$$

in which E is a column matrix involving the n node-pair voltage variables, and I_s is a column matrix with elements equal to the current sources bridged across node pairs (tree branches).

By means of a linear transformation of variables, we wish to introduce the normal coordinates of this system, which means that we wish, with a single transformation, to diagonalize simultaneously the matrices C and G. We can think of accomplishing this end in two separate steps. In the first, we transform C congruently to its canonic form* having all plus one's on the principal diagonal and all other elements zero (C is assumed to be non-singular as well as positive-definite). This transformation affects G also, but leaves it dissymmetrical and non-diagonal.

Denoting the pertinent transformation matrix by A, this step is expressed by the equation

$$A_t \times Y \times A = U_n s + \bar{G} \tag{3}$$

in which A_t is the transpose of A, U_n is a unit matrix of order n, and $\bar{G}$ denotes the resultant form of the conductance matrix.

If $g_1, g_2, \ldots, g_n$ are the latent roots of $\bar{G}$, and B denotes the pertinent modal matrix fulfilling the relation

$$B^{-1} \times \bar{G} \times B = G_d = \begin{bmatrix} g_1 & 0 & 0 & \ldots & 0 \\ 0 & g_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & g_n \end{bmatrix} , \tag{4}$$

then the desired simultaneous diagonalization is expressed by

$$P_t \times Y \times Q = Y_d = \begin{bmatrix} (g_1 + s) & 0 & \ldots & 0 \\ 0 & (g_2 + s) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (g_n + s) \end{bmatrix} \tag{5}$$

*See, for example *The Mathematics of Circuit Analysis* (New York: John Wiley, 1949) pp. 111-118.

where

$$P = A \times B_t^{-1} \qquad \text{and} \qquad Q = A \times B \ . \tag{6}$$

The co-linear transformation expressed by Eq. (4) has no effect upon the canonic form U_n of the capacitance matrix, and so the manipulation (Eq. (5)) achieves the diagonalization of the admittance matrix Y and reveals the latent roots $g_1, g_2, \ldots, g_n$ as being the natural frequencies of the active network. Since the latent roots of a real non-symmetrical matrix are in general complex (but not necessarily so), we see that complex natural frequencies are possible in an RC network with embedded non-bilateral elements.

The response functions of our network are elements of the inverse of the Y matrix, since these characterize the solution to the equilibrium equations (2). From Eq. (5) we have

$$Y^{-1} = Z = Q \times Z_d \times P_t \tag{7}$$

with

$$Z_d = Y_d^{-1} = \begin{bmatrix} \frac{1}{g_1 + s} & 0 & \ldots & 0 \\ 0 & \frac{1}{g_2 + s} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & & \frac{1}{g_n + s} \end{bmatrix} . \tag{8}$$

If $q_{i\nu}$ and $p_{k\nu}$ denote elements in the matrices Q and P (Eq. (6)), we have for the elements of the matrix Z (the open-circuit driving-point and transfer impedances) the expression

$$z_{ik} = \sum_{\nu=1}^{n} \frac{k_\nu}{g_\nu + s} \tag{9}$$

where

$$k_\nu = q_{i\nu}\, p_{k\nu} \ . \tag{10}$$

We recognize Eq. (9) to be the partial fraction expansion of the impedance function z_{ik} and (10) are the residues in its poles. These residues fix only a minor portion of the matrices P and Q and hence only a minor portion of the matrices A and B. We are free to choose the balance of these matrices in whatever manner best suits the realization of the resultant parameter matrices G and C, which are given by

$$C = A_t^{-1} \times A^{-1} = P_t^{-1} \times Q^{-1} \tag{11}$$

and

$$G = A_t^{-1} \times \bar{G} \times A^{-1} = P_t^{-1} \times G_d \times Q^{-1} \quad . \tag{12}$$

Since this conductance matrix represents not only the passive resistance branches but also the embedded active non-bilateral devices (vacuum tubes or transistors), it should be regarded as a sum

$$G = G_1 + G_2 \tag{13}$$

in which G_1 is symmetrical and realizable by an $n+1$-node network with passive elements, while G_2 has the structure and element values pertinent to one or more physical vacuum tubes or transistors connected to the same node pattern. Since this decomposition of G into additive parts is obviously not unique, it injects additional flexibility into the construction of matrices P and Q (or of A and B).

In this connection, it should be observed that while the latent roots of a real dissymmetrical matrix like $\bar{G}$ are usually complex, they need not be and neither need the matrix B be complex. When it is complex, then P and Q in Eq. (6) are, of course, also complex, but one can approach the synthesis problem by constructing either real or complex matrices P and Q. In this regard we have a number of alternative possibilities to consider.

(a) We can construct real matrices P and Q, forming a dissymmetrical matrix G with real latent roots. However, any assumed driving-point or transfer impedance having the partial fraction expansion (9) with residues given by Eq. (10) may have zeros in the complex plane. Hence, a voltage transfer ratio like $E_2/E_1 = z_{12}/z_{11}$ can have zeros and poles in the complex plane. The real latent roots $g_1 \ldots g_n$ of $\bar{G}$, which are poles of z_{11} and z_{12}, have no direct influence upon the transfer ratio E_2/E_1 and may be chosen in any way that facilitates realizability of the pertinent parameter matrices.

(b) We can assume complex poles for a desired transfer impedance z_{12} and prescribe these as the latent roots of $\bar{G}$. The construction of P and Q then involves complex elements since those of the matrix B are complex, although A remains real; and, of course, G, according to Eq. (12), must be real. The assumption of a triangular form for P and Q, and hence for P^{-1} and Q^{-1}, facilitates the construction of these matrices although their complex character adds to the computational labor, particularly where some cut-and-try is involved. However, such difficulties may be resolved with the help of appropriate computing facilities.

III. SUFFICIENCY OF SINGLE-ELEMENT-KIND REALIZATION TECHNIQUES

We shall now consider the realization of networks when their parameter matrices are known. These we assume to be of order n ap-

propriate to equilibrium equations on the node basis for node-pair voltages identified with the branch voltages of a chosen tree in a network graph involving $n+1$ nodes.

Let us consider for a moment that only a single parameter matrix, say G of order n, is specified. The simplest network for its realization has $n+1$ nodes since the pertinent graph must at least have a tree with n branches. This same matrix G may, of course, be realized by a network involving more than $n+1$ nodes. In fact, such a more complicated network may realize a G matrix not realizable with an $n+1$-node graph. As we shall point out presently, however, this more general realization is not acceptable for our purposes, and so only the $n+1$-node realization techniques are pertinent to the problem considered here.

Realization of a single parameter matrix (for example, a G matrix) by an $n+1$-node network has recently been accomplished.* The conditions for its realizability are expressible in a variety of ways, some of which are adaptable to the needs of the methods of matrix construction discussed above. We shall not include the details of these realization procedures here but shall confine our discussion to showing that the $n+1$-node single-element-kind realization process is all that we need for the construction of three-element-kind networks from their parameter matrices pertinent to a node basis.

When a single matrix like G is given and a network realization involving only positive elements and no ideal transformers is asked for, then a necessary condition for the existence of this network is that G be appropriate to some geometrical tree configuration. Let us clarify this statement by reminding the reader that in an analysis problem where the network is given and we are asked to construct G corresponding to a set of equilibrium equations, it is first necessary to choose voltage variables. These may be the branch voltages of a tree, or they may be any linear independent combinations of branch voltages or node-pair voltages. In the latter instance, the resulting G matrix is not based upon a tree; in the former, it is. A necessary condition for a given G matrix to be $n+1$-node realizable without ideal transformers is that it be based upon a tree.**

In the synthesis of active RC networks discussed above, both matrices G and C must be based upon the same geometrical tree configuration (a condition that is easy to meet in the process of matrix construction). Realization of the total network is accomplished

*See, for example, "Applications of Matrix Algebra to Network Theory" by I. Cedarbaum (*IRE Trans. on Circuit Theory*, Vol. CT-6, May 1959) p. 127. A simpler procedure more adaptable to the needs of the present problem is given in the M.I.T. Res. Lab. of Electronics Quarterly Progress Report for Jan. 15, 1960 entitled "On the Synthesis of Single-Element-Kind Networks," by E.A. Guillemin.

**There are simple ways of testing a given G matrix to see if it is based upon a tree. See the M.I.T. RLE QPR reference cited above.

in two steps. First, each matrix is realized by a single-element-kind network for the implied common tree configuration; then the several networks are joined by connecting corresponding tree branches in parallel (placing the trees in parallel by soldering together their corresponding nodes).

The fact that this procedure yields a network with the desired admittance matrix Y is easy to see. Consider, for example, the RC network situation represented by Eq. (1). This is an n terminal-pair network whose Y matrix is given by the sum of two component Y matrices, namely, the matrices Cs and G. If two separate n terminal-pair networks having these component Y matrices are found, then their parallel combination has a resultant Y matrix equal to the sum of the component matrices provided certain conditions assuring no circulating currents resulting from the parallel connection* are fulfilled. It is readily established that these conditions are automatically fulfilled if the n+1-node realizations for the separate matrices C and G are based upon the same geometrical tree configuration.

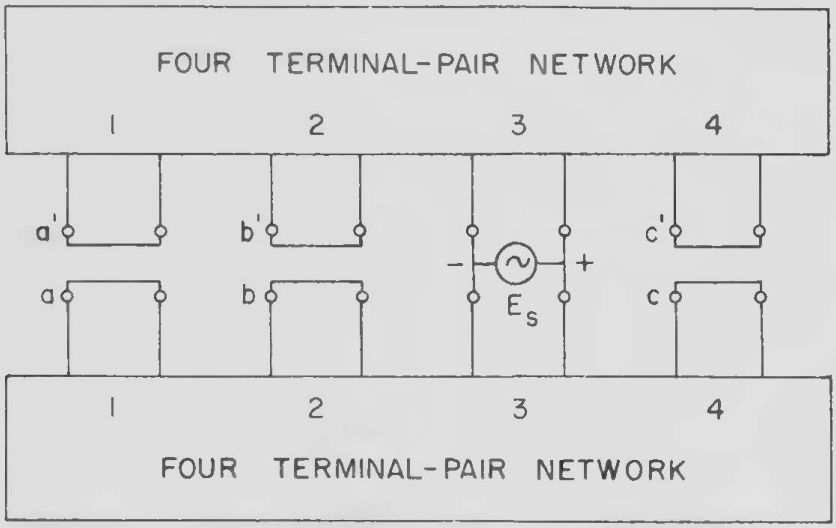

Fig. 1 One of the tests to determine whether the parallel connection of two four terminal-pair networks will cause circulatory currents

The schematic diagram in Fig. 1 illustrates the principle involved here. Straightforward extension of the well-known tests** for two terminal-pair networks, amounts to applying a voltage source E_s to one of the paralleled pairs of terminals as shown in the figure, separately shorting all other terminal pairs, and then testing for zero voltage between those shorted pairs like a-a', b-b' that are to be connected in parallel. This test must be repeated n times with the source E_s across each terminal pair. If the terminal pairs are

*See *Synthesis of Passive Networks* by E.A. Guillemin (New York: John Wiley, 1957) pp. 191-194.

**Ibid

across corresponding branches of identical tree configurations, then their mutual adjacency (which insures a common terminal between pairs) is readily seen to assure that all tests yield zero voltage as required.

Therefore, when we approach the synthesis problem from a parameter matrix point of view rather than a conventional one in which the procedure yields a network configuration directly from the given impedance function, all we need for realization of the resulting network is a technique for constructing a single-element-kind network from a given parameter matrix. It is rather surprising to find that the latter problem, which lay neglected for so many years (perhaps because no one considered it of sufficient practical interest to bother with), should become *the* tool for synthesis of completely general networks with unrestricted topological configuration!

IV. EXAMPLES

Let us say that we want an active RC network that realizes an open-circuit transfer impedance

$$z_{12} = \frac{K}{(s - s_1)(s - s_2)} \quad , \tag{14}$$

in which s_1 and s_2 are the conjugate complex poles

$$\begin{aligned} s_1 &= -\sigma_o - j\omega_o \quad , \\ s_2 &= -\sigma_o + j\omega_o \quad . \end{aligned} \tag{15}$$

If, for the modular matrix B in Eq. (4) we assume

$$B = \begin{bmatrix} 1 & j \\ -j & -1 \end{bmatrix} \quad , \tag{16}$$

having the inverse

$$B^{-1} = \begin{bmatrix} 1/2 & j/2 \\ -j/2 & -1/2 \end{bmatrix} \quad , \tag{17}$$

then we get

$$\bar{G} = B \times \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \times B^{-1} = \begin{bmatrix} 1/2\,(g_1 + g_2) & j/2\,(g_1 - g_2) \\ -j/2\,(g_1 - g_2) & 1/2\,(g_1 + g_2) \end{bmatrix} \tag{18}$$

and if we identify g_1 and g_2, respectively, with $-s_1$ and $-s_2$ in Eq. (15), we have

$$\bar{G} = \begin{bmatrix} \sigma_o & -\omega_o \\ \omega_o & \sigma_o \end{bmatrix} . \tag{19}$$

The open-circuit impedance matrix Z, according to Eq. (7), is given by

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = A \times B \times \begin{bmatrix} \frac{1}{s - s_1} & 0 \\ 0 & \frac{1}{s - s_2} \end{bmatrix} \times B^{-1} \times A_t . \tag{20}$$

Here the matrix A must be chosen so as to yield the desired z_{12} function, on the one hand, and realizable C and G matrices according to Eqs. (11) and (12), on the other.

Regarding realizability conditions, let us assume the simplest situation possible, namely, that for which the parameter matrices are based upon a node-to-datum choice of node-pair voltage variables (corresponding to a star-like tree). These matrices then must have the so-called "dominant" form (diagonal elements are positive, non-diagonal ones are negative, and the algebraic sum of elements in any row is non-negative). According to well-known methods of network analysis for such a node basis, the circuit for parameter matrices in this form can be recognized by inspection.

Although in our present problem we could choose a variety of A matrices to fulfill the necessary conditions, let us pick the simplest one possible, namely, the unit matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{21}$$

Then Eqs. (16), (17) and (20) give

$$z_{11} = z_{22} = \frac{1/2}{s - s_1} + \frac{1/2}{s - s_2} , \tag{22}$$

$$z_{12} = -z_{21} = \frac{j/2}{s - s_1} - \frac{j/2}{s - s_2} ,$$

$$= \frac{\omega_o}{(s - s_1)(s - s_2)} , \tag{23}$$

which is the desired result. The capacitance matrix C is the unit matrix (21) and the conductance matrix G is identical with $\bar{G}$ given by Eq. (19).

According to Eq. (13) we must separate this G matrix into two additive portions, one of which must be dominant, the other representative of some combination of active (and non-bilateral) elements appropriate to the same node basis with star-like tree. Suppose we choose for the active and non-bilateral elements the matrix

$$G_2 = \begin{bmatrix} -g & 0 \\ g_m & -g \end{bmatrix} \tag{24}$$

in which g and g_m are positive constants. (We might, for example, obtain such a matrix from the combination of a pentode and two "tunnel" diodes.)

The passive bilateral component of G then becomes

$$G_1 = G - G_2 = \begin{bmatrix} (\sigma_o + g) & -\omega_o \\ -(g_m - \omega_o) & (\sigma_o + g) \end{bmatrix} . \tag{25}$$

Since this matrix must be symmetrical, we have

$$\omega_o = g_m/2 \tag{26}$$

and its dominant character requires that

$$g \geqslant \omega_o - \sigma_o . \tag{27}$$

These relations fix the matrix of the active element for a desired complex pole, or for a given active element they place limitations upon the complex natural frequency obtainable in this simple circuit. The usual techniques of frequency scaling are, of course, applicable here as they are in passive synthesis procedures, so that the conditions expressed by Eqs. (26) and (27) should be regarded as binding only insofar as they relate to the ratio

$$\sigma_o/\omega_o \geqslant 1 - 2g/g_m . \tag{28}$$

By choosing other possible matrices A, rather than the unit matrix (21), and then decomposing the resultant G in various ways, one can obtain an endless variety of other network combinations of active resistive and passive RC elements, all of which yield the same desired form of transfer impedance.

We will next consider the problem of obtaining essentially the

same transfer function (Eq. (14)) as a voltage ratio. Specifically, we will consider

$$\frac{E_2}{E_1} = \frac{z_{12}}{z_{11}} = \frac{K s^2}{(s - s_a)(s - s_b)} \tag{29}$$

with s_a and s_b given by the right-hand sides of Eqs. (15).

For the open-circuit impedances we assume the forms

$$z_{12} = \frac{K s^2}{(s + g_1)(s + g_2)(s + g_3)} = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \frac{k_3}{s - s_3} , \tag{30}$$

$$z_{11} = \frac{(s - s_a)(s - s_b)}{(s + g_1)(s + g_2)(s + g_3)} = \frac{\rho_1}{s - s_1} + \frac{\rho_2}{s - s_2} + \frac{\rho_3}{s - s_3} . \tag{31}$$

Here the poles determined by values of $g_1 = -s_1$, $g_2 = -s_2$ and $g_3 = -s_3$ have no influence upon the transfer function (29). Their values must, of course, be real and positive but they can be chosen in any way that best suits other conditions of the problem. Suppose we assume their numbering to be such that $g_1 < g_2 < g_3$. Then the residues are

$$k_1 = \frac{g_1^2 K}{(g_1 - g_2)(g_1 - g_3)} > 0 , \tag{32}$$

$$k_2 = \frac{g_2^2 K}{(g_2 - g_1)(g_2 - g_3)} < 0 , \tag{33}$$

$$k_3 = \frac{g_3^2 K}{(g_3 - g_1)(g_3 - g_2)} > 0 . \tag{34}$$

From the fact that $z_{12} \longrightarrow K/s$ for $s \longrightarrow \infty$ we have

$$K = k_1 + k_2 + k_3 . \tag{35}$$

If we denote

$$f(s) = (s - s_a)(s - s_b) , \tag{36}$$

then the residues of z_{11} in Eq. (31) are given by

$$\rho_1 = \frac{k_1 f(s_1)}{K g_1^2} , \qquad \rho_2 = \frac{k_2 f(s_2)}{K g_2^2} , \qquad \rho_3 = \frac{k_3 f(s_3)}{K g_3^2} , \tag{37}$$

and since $z_{11} \rightarrow 1/s$ for $s \rightarrow \infty$ we have

$$\rho_1 + \rho_2 + \rho_3 = 1 . \tag{38}$$

According to Eq. (10) these residues determine the following relations between elements in the matrices P and Q (Eq. (6))

$$\rho_1 = q_{11} p_{11} , \qquad \rho_2 = q_{12} p_{12} , \qquad \rho_3 = q_{13} p_{13} \tag{39}$$

and

$$k_1 = q_{11} p_{21} , \qquad k_2 = q_{12} p_{22} , \qquad k_3 = q_{13} p_{23} . \tag{40}$$

Apart from these relations the matrices A and B determining P and Q in Eq. (6) may be chosen in whatever way meets realizability of the resultant parameter matrices C and G.

These considerations evidently involve not only the matrices A and B but also their respective inverses. In order to facilitate matters, it is expedient to assume a triangular form for these matrices since the inverse of a matrix in this form is more easily written. Thus, suppose we write

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \tag{41}$$

then

$$B^{-1} = \begin{bmatrix} \frac{1}{b_{11}} & \frac{-b_{12}}{b_{11} b_{22}} & \left[\frac{b_{12} b_{23}}{b_{11} b_{22} b_{33}} - \frac{b_{13}}{b_{11} b_{33}} \right] \\ 0 & \frac{1}{b_{22}} & \frac{-b_{23}}{b_{22} b_{33}} \\ 0 & 0 & \frac{1}{b_{33}} \end{bmatrix} \tag{42}$$

and the matrix G, according to Eq. (4), is given by

$$\bar{\bar{G}} = \begin{bmatrix} g_1 & (g_2 - g_1)\frac{b_{12}}{b_{22}} & \left[(g_1 - g_2)\frac{b_{12}\,b_{23}}{b_{22}\,b_{33}} + (g_3 - g_1)\frac{b_{13}}{b_{33}}\right] \\ 0 & g_2 & (g_3 - g_2)\frac{b_{23}}{b_{33}} \\ 0 & 0 & g_3 \end{bmatrix} \tag{43}$$

If we assume for A the same triangular form (41), then its inverse is given by (42) with the letters a and b interchanged. The elements of the matrices P and Q (Eq. (6)) pertinent to expressions (39) and (40) for the residues are then found to be given by

$$q_{11} = a_{11}b_{11}\,, \qquad q_{12} = a_{11}b_{12} + a_{12}b_{22}\,,$$
$$q_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}\,; \tag{44}$$

$$p_{11} = \frac{a_{11}}{b_{11}} - \frac{a_{12}b_{12}}{b_{11}b_{22}} + \frac{a_{13}b_{12}b_{23}}{b_{11}b_{22}b_{33}} - \frac{a_{13}b_{13}}{b_{11}b_{33}}$$
$$p_{12} = \frac{a_{12}}{b_{22}} - \frac{a_{13}b_{23}}{b_{22}b_{33}}\,, \qquad p_{13} = \frac{a_{13}}{b_{33}}\,; \tag{45}$$

and

$$p_{21} = -\frac{a_{22}b_{12}}{b_{11}b_{22}} + \frac{a_{23}b_{12}b_{23}}{b_{11}b_{22}b_{33}} - \frac{a_{23}b_{13}}{b_{11}b_{33}}\,,$$
$$p_{22} = \frac{a_{22}}{b_{22}} - \frac{a_{23}b_{23}}{b_{22}b_{33}}\,, \qquad p_{23} = \frac{a_{23}}{b_{33}}\,. \tag{46}$$

Eqs. (39) are then

$$\rho_1 = a_{11}^2 - \frac{a_{11}a_{12}b_{12}}{b_{22}} + \frac{a_{11}a_{13}b_{12}b_{23}}{b_{22}b_{33}} - \frac{a_{11}a_{13}b_{13}}{b_{33}} \tag{47}$$

$$\rho_2 = a_{12}^2 + \frac{a_{11}a_{12}b_{12}}{b_{22}} - \frac{a_{11}a_{13}b_{12}b_{23}}{b_{22}b_{33}} - \frac{a_{12}a_{13}b_{23}}{b_{33}} \tag{48}$$

$$\rho_3 = a_{13}^2 + \frac{a_{11}a_{13}b_{13}}{b_{33}} + \frac{a_{12}a_{13}b_{23}}{b_{33}} \tag{49}$$

and Eqs. (40) become

$$k_1 = -\frac{a_{11}a_{22}b_{12}}{b_{22}} + \frac{a_{11}a_{23}b_{12}b_{23}}{b_{22}b_{33}} - \frac{a_{11}a_{23}b_{13}}{b_{33}} \tag{50}$$

$$k_2 = a_{12}a_{22} + \frac{a_{11}a_{22}b_{12}}{b_{22}} - \frac{a_{11}a_{23}b_{12}b_{23}}{b_{22}b_{33}} - \frac{a_{12}a_{23}b_{23}}{b_{33}} \tag{51}$$

$$k_3 = a_{13}a_{23} + \frac{a_{11}a_{23}b_{13}}{b_{33}} + \frac{a_{12}a_{23}b_{23}}{b_{33}} \; . \tag{52}$$

The sum of Eqs. (47), (48) and (49), noting (38), yields the condition

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \tag{53}$$

while the sum of Eqs. (50), (51) and (52), noting (35), results in

$$K = a_{12}a_{22} + a_{13}a_{23} \; . \tag{54}$$

These are conditions on the elements of the matrix A.

The ratio of Eq. (49) to Eq. (52) yields a third condition:

$$\frac{a_{13}}{a_{23}} = \frac{\rho_3}{k_3} = \frac{f(s_3)}{K g_3^2} \; . \tag{55}$$

Subtraction of Eq. (47), multiplied by a_{23} from Eq. (50), multiplied by a_{13}, gives

$$\frac{b_{12}}{b_{22}} = \frac{a_{11}^2 a_{23} + a_{13}k_1 - a_{23}\rho_1}{a_{11}(a_{12}a_{23} - a_{22}a_{13})} \; . \tag{56}$$

Adding Eqs. (47) and (48), and making use of Eqs. (38) and (53), we find

$$\frac{b_{13}}{b_{33}} + \frac{a_{12}}{a_{11}}\frac{b_{23}}{b_{33}} = \frac{\rho_3}{a_{11}a_{13}} - \frac{a_{13}}{a_{11}} \quad . \tag{57}$$

Eliminating the b_{13}/b_{33} term between Eqs. (47) and (57) and again making use of Eqs. (38) and (53), we get

$$\frac{b_{23}}{b_{33}} = \frac{a_{12}^2 - \rho_2 + a_{11}a_{12}(b_{12}/b_{22})}{a_{12}a_{13} + a_{11}a_{13}(b_{12}/b_{22})} \quad . \tag{58}$$

The last three equations enable us to compute elements b_{12}, b_{13} and b_{23} in the B matrix in terms of elements of the A matrix. Eqs. (53), (54) and (55) are three constraint relations on the A matrix which contains six elements, and Eqs. (56), (57) and (58) similarly express three constraints upon the B matrix which likewise contains six elements. Altogether these six constraints express the six relations in Eqs. (39) and (40) required by the residues of z_{11} and z_{12}. We are still free to choose three things in each of the matrices A and B.

A computationally expedient procedure is to choose an A matrix consistent with the three conditions (53), (54) and (55), then compute b_{12}/b_{22} from Eq. (56), next b_{23}/b_{33} from Eq. (58), and finally b_{13}/b_{33} from Eq. (57). The matrix $\bar{G}$ can then be calculated from Eq. (43) after one has decided upon appropriate values for the quantities g_1, g_2 and g_3. Equation (12) then yields the conductance matrix G.

We shall not enter here upon an exhaustive investigation of all possibilities that are thus available, but rather conclude our example by some simple choices. To begin with, we observe from Eq. (11) and the triangular form of the A matrix that if, in the inverse of the C matrix (which is AA_t), we choose the 11-element equal to unity, Eq. (53) is fulfilled, and if we choose the 12-element also equal to unity, then Eq. (54) yields $K = 1$ (which is a nice simple choice). Finally, if in this same matrix $C^{-1} = AA_t$ we choose the 13- and 23-elements alike, then Eq. (55) (together with $K = 1$) gives $f(s_3)/g_3^2 = 1$ which, from a pertinent pole-zero plot in the s-plane, looks like a reasonable thing to do.

These choices correspond to a C^{-1} matrix of the form

$$C^{-1} = \begin{bmatrix} 1 & 1 & a \\ 1 & b & a \\ a & a & c \end{bmatrix} \quad . \tag{59}$$

It is a matter of simple algebra to determine that the C matrix will fulfill the dominance conditions if

$$b > 1 \quad \text{and} \quad c \geqslant a^2 < 1 \ . \tag{60}$$

For $a = 1/\sqrt{2}$, $b = 2$, $c = 1$ we get

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

and

$$C = \begin{bmatrix} 3 & -1 & -\sqrt{2} \\ -1 & 1 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix} \tag{62}$$

which is dominant as required.

Next we make the simple choices

$$g_1 = 1, \qquad g_2 = 2, \qquad g_3 = 3 \ . \tag{63}$$

According to Eq. (55) and choices already made, we must have $f(s_3) = g_3^2 = 9$, which yields for $s_a = -\sigma_a + j\omega_a$ the condition

$$\sigma_a^2 + \omega_a^2 = 6\sigma_a \ . \tag{64}$$

We can get any ω_a/σ_a ratio we want from this. We arbitrarily let $\sigma_a = 1/2$ and find $\omega_a = \sqrt{11}/2$. Also

$$f(s_1) = 3, \qquad f(s_2) = 5, \qquad f(s_3) = 9 \ . \tag{65}$$

Since in Eq. (30) for z_{12}, $K = 1$, we get for the residues of this function

$$k_1 = 1/2, \qquad k_2 = -4, \qquad k_3 = 9/2 \tag{66}$$

and for those of z_{11} according to Eqs. (37)

$$\rho_1 = 3/2, \qquad \rho_2 = -5, \qquad \rho_3 = 9/2 \ . \tag{67}$$

Equation (56) now gives

$$b_{12}/b_{22} = \sqrt{2} \tag{68}$$

and Eqs. (57) and (58) then yield

$$b_{23}/b_{33} = \frac{11}{3}\sqrt{3}, \qquad \text{and} \qquad b_{13}/b_{33} = \frac{13}{6}\sqrt{6}\ . \tag{69}$$

The matrix $\overline{G}$ according to Eq. (43) is then readily evaluated to be

$$G = \begin{bmatrix} 1 & 2 & 2 & \frac{2}{3} \\ 0 & 2 & \frac{11}{3} & \\ 0 & 0 & 0 & \end{bmatrix} \tag{70}$$

and Eq. (12) finally yields (after inverting the A matrix as given by Eq. (42))

$$G = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 3\ \ 2 \\ -2 & -2 & 0 \end{bmatrix} \tag{71}$$

There are various ways in which this G matrix may be separated into a passive bilateral component G_1 and an active non-bilateral component G_2. A possible way is to consider one pentode (tube A) with its cathode, grid and plate connected respectively to nodes 1, 2, 3, giving a conductance matrix of the form

$$\begin{bmatrix} g_1 & -g_1 & 0 \\ 0 & 0 & 0 \\ -g_1 & g_1 & 0 \end{bmatrix}, \tag{72}$$

a second one (tube B) with its cathode, grid and plate connected to datum and nodes 2, 1 giving essentially

$$\begin{bmatrix} 0 & g_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{73}$$

A third (tube C) with its cathode, grid and plate connected respectively to datum and nodes 3, 2, and a fourth (tube D) with its cathode, grid and plate connected respectively to nodes 3, 2 and datum. For these two the conductance matrices are given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g_3 \\ 0 & 0 & 0 \end{bmatrix} \quad (74) \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -g_4 & 0 \end{bmatrix} \quad (75)$$

The resulting schematic for this arrangement of pentodes is given in Fig. 2, and the net conductance matrix is the sum of matrices (72), (73), (74) and (75) namely,

$$G_2 = \begin{bmatrix} g_1 & -(g_1-g_2) & 0 \\ 0 & 0 & g_3 \\ -g_1 & (g_1-g_4) & 0 \end{bmatrix} \quad (76)$$

A dominant matrix G_1 added to this one is seen to be capable of yielding the relative numerical values in (71).

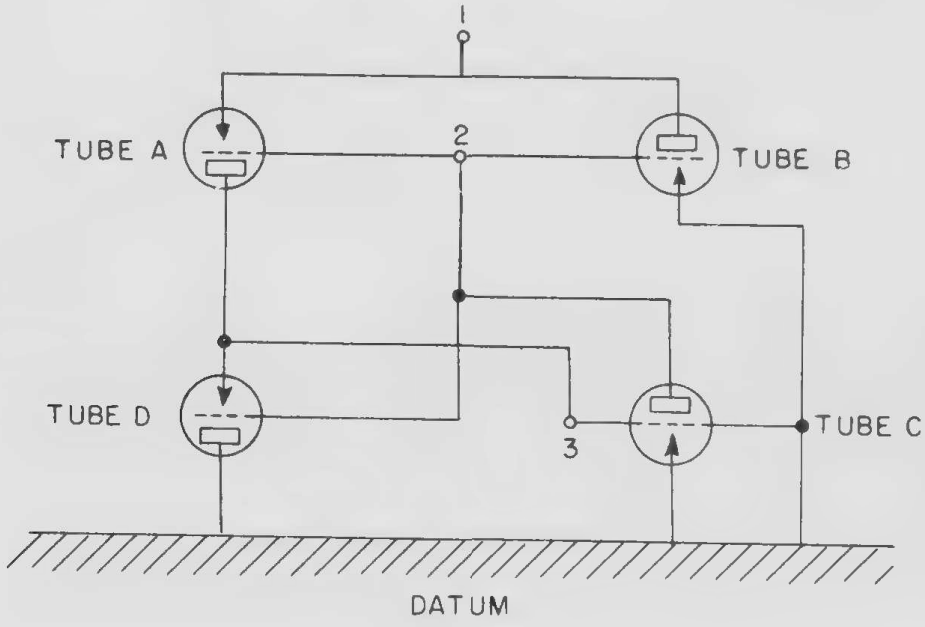

Fig. 2 Schematic arrangement of pentode vacuum tubes which yield the conductance matrix given in Eq.76

By disposing of the several free choices in other ways one can, of course, vary the final requirements considerably, and in any actual problem a careful study of these things should be made. As far as our example here is concerned, however, the essential features of the method are sufficiently illustrated if we terminate our discussions at this point.